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## TWO-DIMENSIONAL MARKOV FIELD AND STATISTICS OF INTEGRAL QUADRATIC FUNCTIONAL BASED ON 2D-FIELD

*Random normal processes and random normal fields possessing the Markov property have widespread applications in various fields, such as communication theory and statistical radiophysics. The object of this paper is to study the statistics of integral quadratic functionals based on Markov processes and fields. The one-dimensional Markovian process of Ornstein-Uhlenbeck is examined. The solid function of the division of the integral quadratic functional based on the one-dimensional normal Ornstein-Uhlenbeck process is considered. An explicit expression for the rigid function and the division of the integral quadratic functional has been extracted. The two-dimensional Markovian 2D-field and the based for the new integral quadratic functional are examined. An analysis of the rigid function of the division of the integral quadratic functional based on a two-dimensional Markovian 2D-field has been carried out. An explicit expression for this integral quadratic functional has been found. We list the general properties of the found solution. From the theory of such functionals, based on solutions of stochastic differential equations, it follows: all zeros of the generating function are simple; the density function in the fluctuation region decays faster than any polynomial and is identically zero at  $\eta = 0$ ; the density function in the peripheral region has an exponential asymptotic behavior with decrement  $\nu$ ; the density function has one maximum and two inflection points; the formula for the autoconvolution of the density function also has the form density function. The above allows us to speak about the Laguerre property of the found probability distribution density. A formalized solution for a stationary normal Markovian rich-atomic field is presented.*

*A further step in this direction is to include a signal component with deterministic properties in the observation functional  $J[H]$ . We note that complete information about the probability distribution density of the random variable  $J[H]$  under consideration makes it possible to successfully resolve evaluation and decision-making problems based on statistical criteria.*

**Keywords:** *one-dimensional Ornstein-Uhlenbeck normal process, two-dimensional Ornstein-Uhlenbeck normal process, Markovian 2D-field, integral quadratic functional, basis on a two-dimensional Markov 2D-field, a solid function, an analytical representation for the required expression of the integral quadratic functional, a well-defined solution for the stationary normal Markovian rich-dimensional field.*

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## ДВУВИМІРНЕ МАРКІВСЬКЕ ПОЛЕ І СТАТИСТИКА ІНТЕГРАЛЬНОГО КВАДРАТИЧНОГО ФУНКЦІОНАЛУ, ЗАСНОВАНОГО НА 2D-ПОЛІ

*Випадкові нормальні процеси та випадкові нормальні поля, що мають марківську властивість, мають найширше застосування в різних галузях, наприклад, у теорії зв'язку та статистичної радіофізики. Об'єктом даної роботи є вивчення статистики інтегральних квадратичних функціоналів, заснованих на марківських процесах і полях. Розглянуто одномірне марковський процес Орнштейна-Уленбека. Розглянута твірна функція розподілу інтегрального квадратичного функціоналу, заснованого на одномірному марківському процесі Орнштейна-Уленбека. Отримано явний вираз для твірної функції та розподілу цього інтегрального квадратичного функціоналу. Розглянуто двовимірне марківське 2D-поле і заснований на ньому інтегральний квадратичний функціонал. Проведено аналіз твірної функції розподілу інтегрального квадратичного функціоналу, заснованого на двовимірному марківському 2D-полі. Отримано явний вираз для розподілу цього інтегрального квадратичного функціоналу. Перерахуємо загальні властивості знайденого розв'язку. З теорії таких функціоналів, заснованої на розв'язках стохастичних диференціальних рівнянь, впливає: усі нулі твірної функції є простими; функція густини в області флуктуації спадає швидше за будь-який поліном і тотожно дорівнює нулю при  $\eta = 0$ ; функція густини в периферійній області має експоненціальну асимптотичну поведінку з декрементом  $\nu$ ; функція густини має один максимум і дві точки перегину; формула для автозгортки функції густини також має вигляд функції густини. Вищезазначене дозволяє говорити про властивість Лагерра знайденої щільності розподілу ймовірностей. Представлено узагальнення рішення для стаціонарного нормального марківського багатовимірного поля.*

*Подальшим кроком у цьому напрямку є включення сигнальної складової з детермінованими властивостями до функціонала спостереження  $J[H]$ . Зазначимо, що повна інформація про щільність розподілу ймовірностей розглянутої випадкової величини  $J[H]$  дозволяє успішно вирішувати задачі оцінювання та прийняття рішень на основі статистичних критеріїв.*

**Ключові слова:** *одновимірний марківський процес Орнштейна-Уленбека, двовимірний марківський процес Орнштейна-Уленбека, марковське 2D-поле, інтегральний квадратичний функціонал, заснований на двовимірному марковському 2D-полі, твірна функція, аналітичне зображення для шуканого виразу інтегрального квадратичного функціоналу, узагальнення рішення для стаціонарного нормального марківського багатовимірного поля.*

**Analysis of recent studies and publications**

Two-dimensional random fields have long been a subject of study and are used in a wide variety of fields [1–3]. Of the wide variety of possible variants and models of two-dimensional random fields, the real normal Markov field (NMF)  $H(x, y)$  [4–7] is perhaps the most commonly used, providing a convenient object for analysis.

**Purpose of the research**

The object of this paper is to study the statistics of integral quadratic functionals based on Markov processes and fields.

**Problem Statement. Presentation of the main research material**

1. The object of this work is the two-dimensional Ornstein-Uhlenbeck process (OU process), a Markov 2D field, and the statistical properties of the integral quadratic functional  $\eta$ , which is based on a real normal two-dimensional Markov 2D field (NMF-field)  $H(x, y)$ . In a detailed study of the properties of such objects as, for example, a spatial image on a given rectangle  $[(0, a) \times (0, b)]$ , one of the main questions of estimation theory about the distribution of the values of the functional

$$J[H] = \int_0^a \int_0^b h^2(x, y) dx dy, \tag{1}$$

where  $h = h(x, y)$  is the realization of a two-dimensional Gaussian field  $H(x, y)$  in a rectangular region  $[(0, a) \times (0, b)]$  on the plane. Important in applied terms is the example of a real random NMF field, any orthogonal sections of which have the properties of a stationary Ornstein-Uhlenbeck process [1, 2] (OU process), which in the physics literature is sometimes called a normal Markov process [3]. The defining property of the stationary field under consideration is its correlator

$$K_{XY} \equiv K_{XY}(x, x', y, y') = E_H [h(x, y)h(x', y')] = pq\sigma_H^2, \tag{2}$$

$$p = \exp(-\nu|x - x'|), \quad q = \exp(-\mu|y - y'|),$$

where  $\sigma_H^2 = E_H [h^2(x, y)]$  is the field intensity,  $\nu$  and  $\mu$  are the attenuation decrements of the field intensity along the  $X$  and  $Y$  axes, respectively.

The following transition density for the NMF field can serve as a generalization of the constructions of conditional transition probability densities for the OU process.

$$f_H((h(x, y)|h(x', y), h(x, y'), h(x', y'))) = \frac{1}{\sqrt{2\pi(1-p^2)(1-q^2)\sigma_H^2}} \exp\left(-\frac{A^2(x, y; x', y')}{2\pi(1-p^2)(1-q^2)\sigma_H^2}\right), \tag{3}$$

where  $A(x, y; x', y') = h(x, y) - ph(x', y) - qh(x, y') + pqh(x', y')$ .

For  $x \rightarrow -\infty$  or  $y \rightarrow -\infty$ , we obtain boundary expressions for the partial transition probability densities

$$f_H((h(x, y)|h(x', y))) = \frac{1}{\sqrt{2\pi(1-p^2)\sigma_H^2}} \exp\left(-\frac{(h(x, y) - h(x', y))^2}{2(1-p^2)\sigma_H^2}\right), \tag{4a}$$

$$f_H((h(x, y)|h(x, y'))) = \frac{1}{\sqrt{2\pi(1-q^2)\sigma_H^2}} \exp\left(-\frac{(h(x, y) - h(x, y'))^2}{2(1-q^2)\sigma_H^2}\right), \tag{4b}$$

which are transition densities for partial OU-processes, and for  $x \rightarrow -\infty$  and  $y \rightarrow -\infty$ , we obtain the vertex probability distribution density of the form

$$f_H(h(x, y)) = \frac{1}{\sqrt{2\pi\sigma_H^2}} \exp\left(-\frac{h^2(x, y)}{2\sigma_H^2}\right) \tag{5}$$

for a random variable – a realization of the NMF field at a point with coordinates  $(x, y)$ , and therefore, from relations (3–5), (2) follows.

2. Due to the positive definiteness and additivity of the two-dimensional integral  $\mathcal{J}[H]$  (1) on a rectangle  $[(0, a) \times (0, b)]$ , it is convenient to describe the properties of the functional  $\mathcal{J}[H]$  using a generating function in the form of the following mathematical expectation

$$Q_{XY}(\lambda) = E_H[\exp(-\lambda\mathcal{J}[H])], \tag{6}$$

where  $\lambda$  is the generating parameter.

The desired generating function can be represented as an absolutely convergent product [4]

$$Q_{XY}(\lambda) = \prod_{n=1}^{\infty} \prod_{m=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_{X,n} \lambda_{Y,m}}\right)^{-1/2}, \tag{7}$$

where  $\{\lambda_{X,n}\}$  and  $\{\lambda_{Y,m}\}$  are the sets of eigenvalues of the partial correlators  $K_X$  and  $K_Y$ , associated with the solutions related to the boundaries of the rectangle of integral equations

$$\varphi_X(x) = \lambda_X (K_X, \varphi_X(x)) = \lambda_X \int_0^a \exp(-\nu |x - x'|) \varphi_X(x') dx', \tag{8a}$$

$$\varphi_Y(y) = \lambda_Y (K_Y, \varphi_Y(y)) = \lambda_Y \int_0^b \exp(-\mu |y - y'|) \varphi_Y(y') dy'. \tag{8b}$$

In the case under consideration of a real NMF field for partial generating functions  $Q_X(\lambda)$  and  $Q_Y(\lambda)$  it is known [4] that

$$Q_X(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_{X,n}}\right)^{-1/2} = \left(\frac{4\nu r_X \exp(\nu a)}{\alpha_+^2 \exp(\rho_X a) - \alpha_-^2 \exp(-\rho_X a)}\right)^{1/2}, \tag{9a}$$

where  $r_X = \sqrt{\nu^2 + 2\lambda\nu\sigma_H^2}$ ,  $\alpha_{\pm} = r_X \pm \nu$ , and also

$$Q_Y(\lambda) = \prod_{m=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_{Y,m}}\right)^{-1/2} = \left(\frac{4\mu r_Y \exp(\mu b)}{\beta_+^2 \exp(\beta_+ b) - \beta_-^2 \exp(-\beta_- b)}\right)^{1/2}, \tag{9b}$$

where  $r_Y = \sqrt{\mu^2 + 2\lambda\mu\sigma_H^2}$ ,  $\beta_{\pm} = r_Y \pm \mu$ .

The simple zeros of the denominators of the reduced generating functions  $Q_X(\lambda)$  and  $Q_Y(\lambda)$  are negative-definite sets of numbers  $\{\lambda_{X,n}\}_{n=1}^{\infty}$  and  $\{\lambda_{Y,m}\}_{m=1}^{\infty}$ . Thus, based on these sets (or, more precisely, the rule defining them), it is necessary to construct a formula for the desired generating function. In other words, it is necessary to contract the product in (7) and obtain a constructive analytical expression for  $Q_{XY}(\lambda)$  that does not require finding specific sets of  $\{\lambda_X\}$  and  $\{\lambda_Y\}$ .

The main result of this paper is the following representation for  $Q_{XY}(\lambda)$ :

$$\ln Q_{XY}(\lambda) = \frac{1}{\pi^2} \int_C dx \int_C dy \left(\frac{d}{dx} \ln Q_X(\lambda)\right) \left(\frac{d}{dy} \ln Q_Y(\lambda)\right) \ln \left(1 - \frac{\lambda}{xy} \frac{\langle \eta \rangle_{XY}}{\langle \eta \rangle_X \langle \eta \rangle_Y}\right), \tag{10}$$

where  $\langle \eta \rangle_x = a\sigma_H^2$ ,  $\langle \eta \rangle_y = b\sigma_H^2$ ,  $\langle \eta \rangle_{XY} = ab\sigma_H^2$  is the mean value of functional (1),  $C$  – is the integration contour in the complex plane, passing through the imaginary axis with a semicircle of sufficiently small radius around the point  $(0,0)$  on the left.

The density distribution  $f_J(\eta)$  of random values  $\eta$  of the functional  $J[H]$  is determined by the inverse Laplace transform of the function  $Q_{XY}(\lambda)$ . Such a transformation for functions (9) or (10) can only be performed numerically; therefore, the more accurate the available information about the transform function  $Q_{XY}(\lambda)$ , the more successful this procedure will be.

3. We construct a proof scheme for result (10) based on the well-known canonical Karhunen-Loève expansion [2, 3] over the set of eigenfunctions  $\{\varphi_{XY}(x, y)\}$ , that satisfy equation (8), but with kernel  $K_{XY}$ ,

$$\varphi_{XY}(x, y) = \lambda_{XY} (K_{XY}, \varphi_{XY}(x, y)) = \lambda_{XY} \int_0^a \int_0^b \exp(-\nu |x - x'| - \mu |y - y'|) \varphi_{XY}(x', y') dx' dy' \quad (11)$$

and corresponds to it with the set of eigenvalues  $\{\lambda_{XY}\}$ . The integral equation (11) can be associated with its differential analog

$$\left( \frac{\partial^2}{\partial x^2} - \nu^2 \right) \left( \frac{\partial^2}{\partial y^2} - \mu^2 \right) \varphi_{XY}(x, y) = 4\lambda_{XY} \nu \mu \sigma_H^2 \varphi_{XY}(x, y). \quad (12)$$

Due to the factorization of the kernel  $K_{XY}$  and the commutation of differential operators in (12), the eigenfunction factorization  $\varphi_{XY}(x, y) = \varphi_X(x)\varphi_Y(y)$  into its partial components follows. Therefore, the spectrum  $\{\lambda_{XY}\}$  is the direct product of the spectra  $\{\lambda_X\}$  and  $\{\lambda_Y\}$ , and equation (11) is equivalent to the equation

$$\begin{aligned} \varphi_X(x)\varphi_Y(y) &= \sigma_H^2 \lambda_X (K_X, \varphi_X(x)) \lambda_Y K_Y, \varphi_Y(y) = \\ &= \sigma_H^2 \lambda_X \int_0^a \exp(-\nu |x - x'|) \varphi_X(x') dx' \lambda_Y \int_0^b \exp(-\mu |y - y'|) \varphi_Y(y') dy'. \end{aligned} \quad (13)$$

To obtain the spectrum of result (13), we note that for an arbitrary function, which: a) has a countable set of simple negative zeros  $\{z_k\}$ , b) has an absolutely convergent Hadamard factorized representation with zero growth rate:  $\psi(z) = \prod_{k=1}^{\infty} (1 - z/z_k)^{-1}$ , the equality holds

$$\ln \psi(\lambda) = \frac{1}{2\pi i} \int_C \left( \frac{d}{dz} \ln \psi(z) \right) \ln(1 - \lambda/z_k)^{-1} dz. \quad (14)$$

Then, applying this equality for  $\varphi_X(x)$  and for  $\varphi_Y(x)$  we obtain the desired relation (10), which after renormalization we write in its final form

$$\ln Q_{XY}(\lambda) = -\frac{\lambda}{(\pi i)^2} \int_C dx \int_C dy (xy\sigma_H^2 - \lambda)^{-2} \ln Q_X(x) \ln Q_Y(y). \quad (15)$$

Thus, information about the behavior of the functional  $J[H]$  (1) on the boundaries of a rectangular region for a stationary NMF-field is sufficient to determine its statistics inside this region.

Based on generating function (15), we can find expressions for the moments of the random variable  $J[H]$ . By calculating the first and second  $\lambda$ -derivatives of function (15) at the point  $\lambda = 0$  and determining the partial cumulants, we determine the first moment

$$E_H(J[H]) = ab\sigma_H^2 \equiv \langle \eta \rangle_{XY}, \quad (16)$$

and the second moment, which yields the variance of functional (1).

$$E_H(J^2[H]) - E_H^2(J[H]) = \frac{\langle \eta \rangle_{XY}^2}{4\nu^2 \mu^2 a^2 b^2} (-1 + 2\nu a + e^{-2\nu a}) (-1 + 2\mu b + e^{-2\mu b}). \tag{17}$$

4. Result (10) admits of various generalizations. In particular, it can be extended from the two-dimensional to the multidimensional case. For example, for  $D = 3$ , assuming that, according to (9), for a parallelepiped  $\{x \in (0, a), y \in (0, b), z \in (0, c)\}$  on three Cartesian axes, the partial generating functions  $Q_X(x)$ ,  $Q_Y(y)$  and  $Q_Z(z)$  are given, we find

$$\ln Q_{XYZ}(\lambda) = -\frac{\lambda}{(\pi i)^3} \int_0^a dx \int_0^b dy \int_0^c dz \left( \frac{\partial}{\partial x} \ln Q_X(x) \right) \left( \frac{\partial}{\partial y} \ln Q_Y(y) \right) \left( \frac{\partial}{\partial z} \ln Q_Z(z) \right) \ln \left( 1 - \frac{\lambda}{xyz \sigma_H^2} \right). \tag{18}$$

Here,  $\sigma_H^2$  is the average intensity of the 3D-field. Similar to (18), generating functions can be constructed for normal Markov fields of higher dimensions.

Similar to generating function  $Q_X(x)$  (9), which corresponds to a one-dimensional field (stationary process), expressions like (10) or (18) can be used to process and interpret data on two-dimensional and three-dimensional objects – random stationary Markov fields.

5. Analytical reconstruction of the probability distribution density  $f_J(\eta)$  of random values  $\eta$  of a functional  $J[H]$  presents significant difficulties even in the one-dimensional case [4, 5]. To circumvent the difficulties associated with root uncertainties in the one-dimensional case, we proceed to finding the density  $f_J(\eta)$ , the expression for which we write in terms of the inverse Laplace transform

$$f_J(\eta) = \frac{1}{2\pi i} \int_C \exp\left(\lambda \eta - \xi_x + \sqrt{1 + 2\lambda \langle \eta \rangle_x / \xi}\right) d\lambda, \tag{19}$$

where

$$\xi_x = \frac{2\nu^2 a^2}{-1 + 2\nu a + e^{-2\nu a}}, \quad \langle \eta \rangle_x = a\sigma_H^2. \tag{20}$$

In contour integral (19), we move to the integration variable  $\rho = \sqrt{1 + 2\lambda \langle \eta \rangle_x / \xi}$ . Due to the absence of singular points, the integration contour remains the Bromwich contour, which allows us to transform (20) to the form

$$f_J(\eta) = \frac{\langle \eta \rangle_x}{\xi_x} \int_C \frac{\rho d\rho}{2\pi i} \exp\left(\frac{\rho^2 - 1}{2\xi_x} \langle \eta \rangle_x \eta + \xi_x - \xi_x \rho\right). \tag{21}$$

By forming a perfect square with respect to  $\rho$  in the exponent and using the saddle-point method for integration, we find the value of integral (21):

$$f_J(\eta) = \sqrt{\frac{\xi_x \langle \eta \rangle_x}{2\pi \eta^3}} \exp\left[-\frac{\xi_x}{2} \left(\sqrt{\frac{\langle \eta \rangle_x}{\eta}} - \sqrt{\frac{\eta}{\langle \eta \rangle_x}}\right)^2\right], \quad 0 \leq \eta < \infty. \tag{22}$$

The approximate formula (22) found for the distribution density in the one-dimensional case of random values  $\eta$  describes the contribution of fluctuations of the OU-process along one coordinate  $x$ . A similar procedure (18)–(22) can be used for the OU-process along the axis  $y$ .

We list the general properties of the found solution. We will analyze expression (22) using the framework of quadratic integral functionals outlined in the textbook [5]. From the theory of such functionals, based on solutions of stochastic differential equations, it follows:

- 1) all zeros of the generating function (13) are simple;

2) the density function  $f_j(\eta)$  in the fluctuation region ( $\eta \ll \langle \eta \rangle_X$ ) decays faster than any polynomial and is identically zero at  $\eta = 0$ ;

3) the density function  $f_j(\eta)$  in the peripheral region ( $\eta \gg \langle \eta \rangle_X$ ) has an exponential asymptotic behavior with decrement  $\nu$ ;

4) the density function  $f_j(\eta)$  has one maximum and two inflection points;

5) the formula for the autoconvolution of the density function  $f_j(\eta)$  also has the form (22).

The above allows us to speak about the Laguerre property [6, 7] of the found probability distribution density  $f_j(\eta)$ .

6. Let's move on to finding the generating function  $Q_{XY}(\lambda)$  and consider expression (18). Expanding the last factor in (18) into a series and grouping the terms, we write

$$\frac{d}{d\lambda} \ln Q_{XY}^2(\lambda) = -\frac{1}{(2\pi i)^2} \frac{(-1)^n}{\sigma_H^{2n+2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sigma_H^{2n+2}} \int_c \frac{d^{n+1}}{dx^{n+1}} \left[ \frac{d}{dx} \ln Q_X^2(x) \right] \int_c \frac{d^{n+1}}{dy^{n+1}} \left[ \frac{d^{n+1}}{dy^{n+1}} \ln Q_Y^2(y) \right]. \quad (23)$$

According to Cauchy's integral theorem, this expression can be written after integration as follows:

$$\frac{d}{d\lambda} \ln Q_{XY}^2(\lambda) = -\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!n!\sigma_H^{2n+2}} \left[ \frac{d^{n+1}}{dx^{n+1}} \ln Q_X^2(x) \right] \left[ \frac{d^{n+1}}{dy^{n+1}} \ln Q_Y^2(y) \right]_{x=y=0}, \quad (24)$$

that is, the desired generating function is expressed in terms of the cumulants of the partial generating functions.

Now let's perform the following steps in (24). Now, in the notation for the partial generating functions (12), we restrict ourselves to the leading (exponential) terms:

$$Q_X^2(x) \approx \exp\left(av - \sqrt{v^2 + 2vx\sigma_H^2}\right), \quad Q_Y^2(y) \approx \exp\left(b\mu - \sqrt{\mu^2 + 2\mu y\sigma_H^2}\right). \quad (25)$$

This approximation corresponds to the case when  $av \gg 1$  and  $bv \gg 1$ . Second, we move on to the variables  $u = x\sigma_H^2/v$  and  $v = y\sigma_H^2/\mu$ . Then we obtain

$$\frac{d}{d\lambda} \ln Q_{XY}^2(\lambda) = -ab\sigma_H^2 \sum_{n=0}^{\infty} \frac{(-\lambda\sigma_H^2)^n}{n!n!(\nu\mu)^2} \left[ \frac{d^{n+1}}{du^{n+1}} \sqrt{1+2u} \right] \left[ \frac{d^{n+1}}{dv^{n+1}} \sqrt{1+2v} \right]_{u=v=0}. \quad (26)$$

Let's introduce the notations

$$\xi = avb\mu, \quad \langle \eta \rangle_{XY} = ab\sigma_H^2, \quad (27)$$

with which we will concentrate the fluctuation parameters corresponding to each axis. We differentiate (26) once with respect to  $u$  and once with respect to  $v$ :

$$\frac{d}{d\lambda} \ln Q_{XY}^2(\lambda) = -\langle \eta \rangle \sum_{n=0}^{\infty} \frac{(-\lambda \langle \eta \rangle_{XY} / \xi)^n}{n!n!} \left[ \frac{d^n}{du^n} \frac{1}{\sqrt{1+2u}} \right] \left[ \frac{d^n}{dv^n} \frac{1}{\sqrt{1+2v}} \right]_{u=v=0}. \quad (28)$$

The parameter  $\xi$  determines the magnitude of the dispersion of the functional (1), the expression for which is given by the expression (20) in the one-dimensional case.

To improve the approximation, we will further understand this parameter  $\xi$  as the quantity

$$\xi_{XY} = \frac{2\nu^2 a^2}{-1 + 2\nu a + e^{-2\nu a}} \frac{2\mu^2 b^2}{-1 + 2\mu b + e^{-2\mu b}}, \quad (29)$$

which is associated with the correct value of the variance  $\mathbf{E}_H(J^2[H]) - \mathbf{E}_H^2(J[H])$  for any values  $\nu a$  and  $\mu b$ . Now in (28) for the radical fractions we use the integral representation, take all the

derivatives and sum over  $n$  using the power expansion for the Bessel function, after which we obtain the expression

$$\frac{d}{d\lambda} \ln Q_{XY}^2(\lambda) = -\frac{\langle \eta \rangle}{\pi} \int_0^\infty dt_1 \int_0^\infty dt_2 \exp(-t_1^2 - t_2^2) J_0(4t_1 t_2 \sqrt{\lambda \langle \eta \rangle / \xi_{XY}}). \tag{30}$$

From (30) we can already see the interaction of the fluctuations of the NMF field in both coordinates. Moving on to polar coordinates  $t_1 = \rho \sin \varphi$  and  $t_2 = \rho \cos \varphi$ , we find as a result of integrating over the radius  $\rho$

$$\frac{d}{d\lambda} \ln Q_{XY}^2(\lambda) = \frac{\langle \eta \rangle_{XY}}{2\pi} \int_0^{2\pi} d\varphi \left[ 1 + \frac{4\lambda \langle \eta \rangle_{XY}}{\xi_{XY}} \sin^2 2\varphi \right]^{-1/2}. \tag{31}$$

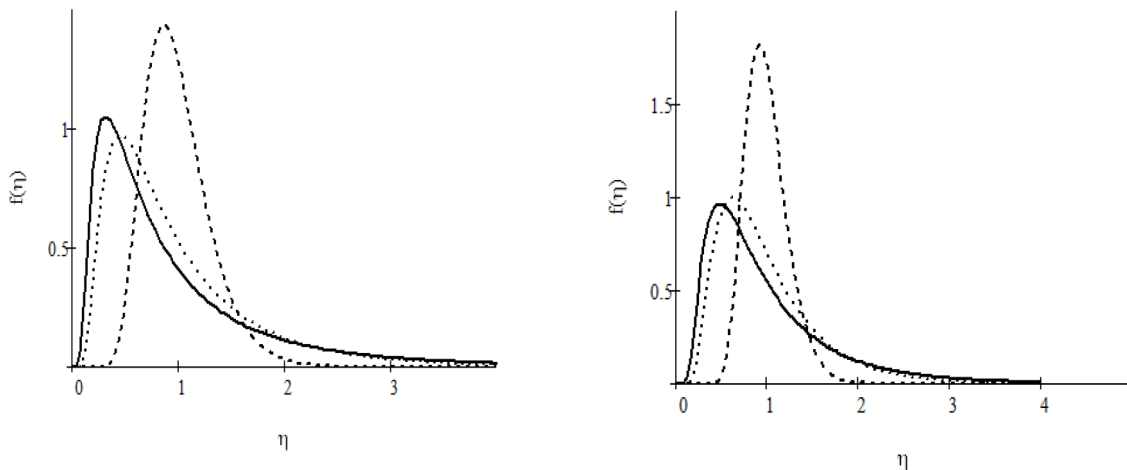
This expression reduces to elliptic integrals. In order to obtain the generating function  $Q_{XY}(\lambda)$  suitable for the inverse Laplace transform, we now use another approximation: in (31), we replace the function  $\sin^2(2\varphi)$  with its average value over the interval  $[0, 2\pi]$ . Then we finally find

$$Q_{XY}^2(\lambda) = \exp\left(\xi_{XY} - \xi_{XY} \sqrt{1 + 2\lambda \langle \eta \rangle_{XY} / \xi_{XY}}\right). \tag{32}$$

Based on (32), after integration using the saddle-point method, we obtain the distribution density of the integral quadratic functional based on a two-dimensional Markovian 2D-field:

$$f_J(\eta) = \sqrt{\frac{\xi_{XY} \langle \eta \rangle_{XY}}{2\pi\eta^3}} \exp\left[-\frac{\xi_{XY}}{2} \left(\sqrt{\frac{\langle \eta \rangle_{XY}}{\eta}} - \sqrt{\frac{\eta}{\langle \eta \rangle_{XY}}}\right)^2\right], \quad 0 \leq \eta < \infty. \tag{33}$$

Note that the expression found (32) is identical in structure to its partial analogue (25). In this regard, it can be hypothesized that the generating functions of quadratic functionals for normal Markov fields of higher dimensions have a similar form.



**Fig. 1. Distribution densities of the integral functional (1).**  
**Parameters:  $a = 1, b = 1, \sigma_H^2 = 1.$**   
**Left:  $\nu = 0.1$  (line);  $\nu = 1$  (dots);  $\nu = 10$  (dash);  $\mu = 0$**   
**Right:  $\nu = 0.1$  (line);  $\nu = 1$  (dots);  $\nu = 1$  (dash);  $\mu = 1$**

Figure 1 shows two groups of probability distribution densities  $f_J(\eta)$  according to (22) and (32). Left  $\mu = 0$ ; Right  $\mu = 1$ .

From the shape of the curves, we can conclude that the densities have the Laguerre property.

### Conclusions

This paper examines a random stationary NMF-field, studying its statistical properties and the properties of the integral quadratic functional defined on it. The solutions found, in terms of estimation theory, relate to the case of recording only the noise component.

A further step in this direction is to include a signal component with deterministic properties in the observation functional  $J[H]$ . Considering this more general case will make it possible to analytically describe the synthesis of optimal receivers recording real-valued two-dimensional signals against a background of real-valued two-dimensional normal Markov noise. Similar to the generating function  $Q_{XY}(\lambda)$  (9), corresponding to a one-dimensional field (stationary process), expression like (32) can be used in processing and interpreting data on two-dimensional and three-dimensional objects. In conclusion, we note that complete information about the probability distribution density of the random variable  $J[H]$  under consideration makes it possible to successfully resolve evaluation and decision-making problems based on statistical criteria.

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