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SYSTEMS OF KINETIC POPULATION EQUATIONS AND THEIR APPLICATIONS

A systematic analysis of systems of kinetic population equations is presented, including Volterra and Lotka-Volterra. Population problems that had to be solved are investigated, and their brief analysis is provided. These problems include demographic, ecological, etc. problems. From a conceptual point of view, these problems are divided into two types: the problem of two species eating the same food (Volterra equation) and the predator-prey problem (Lotka – Volterra equation). The first problem arose from the problem of rabbit reproduction in Australia. In addition, in the same population biology, the problem arose when one species eats another (predator and prey). This problem was solved by many researchers in the field of biology and medicine, in particular virology. Its partial solution is given in the book of A. Lotka, and a more general one in the lectures of V. Volterra. Because of this, these equations are sometimes called the Lotka-Volterra equations. As in the first and second problems, it is necessary that there is enough resource (food) for the stationary stable existence and development of the dynamical system. We have analyzed the problems that are solved or that are expedient to be solved using these methods. Problems with a non-uniform temporal hierarchy of processes have also been analyzed. It has been shown that for solving such problems it is expedient to use the method of adiabatic elimination of variables. This method was used to solve kinetic problems in relaxation optics. These equations are expedient to use when there are several competing in-phase processes. Based on the general analysis of the systems of Volterra equations, it is possible to construct system criteria for controlling and predicting the corresponding processes and phenomena. To move to spatial problems, it is necessary to introduce the corresponding transport and diffusion coefficients into the systems of equations of Volterra and Lotka – Volterra. In this case, these equations can also be considered as systems of nonlinear diffusion equations. A list of problems for which it is expedient to use such a formalism is given.

Key words: *dynamical processes, Volterra, Lotka, adiabatic exclusion, diffusion expansion, population problems, nonlinear dynamics.*

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СИСТЕМИ КІНЕТИЧНИХ ПОПУЛЯЦІЙНИХ РІВНЯНЬ ТА ЇХ ЗАСТОСУВАННЯ

Наводиться системний аналіз систем кінетичних популяційних рівнянь, зокрема й Вольтерри та Лотки – Вольтерри. Досліджені популяційні завдання, які необхідно було розв'язати, та короткий їх аналіз. До цих завдань належать демографічні, екологічні й інші проблеми. З понятійного погляду ці завдання розбиваються на два типи: завдання про два види, що їдять одну їжу (рівняння Вольтерри) та завдання хижак – жертва (рівняння Лотки –Вольтерри). Перше завдання зумовлене проблемою розмноження кроликів в Австралії. Окрім того, у тій же популяційній біології постало завдання, коли один вид поїдає інший (хижак і жертва) Це завдання розв'язувалась багатьма дослідниками в галузі біології та медицини, зокрема вірусології. Її частинний розв'язок наведений у книзі А. Лотки, а більш загальний – у лекціях В. Вольтерри. Тому ці рівняння інколи називають рівняннями Лотки – Вольтерри. Як у першому, так і у другому завданні необхідно, щоб було вдосталь ресурсу (їжі) для стаціонарного стабільного існування та розвитку динамічної системи. Нами провендено аналіз проблем, які розв'язуються або які доцільно розв'язувати за допомогою цих методів. Також проаналізовані задачі з неоднорідною часвою ієрархією процесів. Показано, що для розв'язання таких задач доцільно використовувати метод адіабатичного виключення змінних. Цей метод був використаний для розв'язання кінетичних проблем у релаксаційній оптиці.

Ці рівняння доцільно використовувати тоді, коли є декілька конкуруючих синфазних процесів. На основі загального аналізу систем рівнянь Вольтерри можна побудувати системні критерії управління та прогнозування відповідних процесів і явищ. Для переходу до просторових задач у системі рівнянь Вольтерри та Лотки – Вольтерри потрібно ввести відповідні коефіцієнти переносу та дифузії. У такому разі ці рівняння можна також розглядати як системи нелінійних рівнянь дифузії. Наводиться випадок дифузійної нестійкості.

Ключові слова: *динамічні процеси, Вольтерра, Лотка, адіабатичне виключення, дифузійна нестійкість, популяційні проблеми, нелінійна динаміка.*

Problem Statement

For modeling dynamic (chronological) processes systems of population equations [1], including Volterra [2] and Lotka – Volterra [3] equations, are used. These equations should be used when

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there are several competing in-phase processes. Based on the general analysis of Volterra's equation systems, it is possible to construct systemic criteria for controlling and predicting the corresponding processes and phenomena [4].

Analysis of recent studies and publications

Let us present the simplest system of two equations. Depending on the conditions of the problem, we will consider and analyze problems of four types: the problem of two species eating the same food [2]; the predator-prey problem [2]; the problem with adiabatic elimination [5] of a variable and the problem with diffusion instability [1].

It is shown that the method of aliabatic elimination of variables [5] is one of the main methods of the theory of dissipative structures [6] and synergetics [5].

The problem of diffusion instability for systems of autonomous equations with the addition of diffusion terms is investigated. In general, in a particular case, this can be considered as a diffusion extension of the Volterra equations[1]. The analysis is carried out for a system of two equations.

The feasibility of using these methods to describe various evolutionary dynamic processes of the population type is shown [1].

Presentation of the main research material

Two species eating the same food. In the second half of the XIXth century, the problem of rabbits arose in Australia, which proved to be worthy competitors for farmers. And now their number fluctuates between 0,6 and 0,7 billion. This problem was first solved by Vito Volterra and published at the end of the 19th century in the journal Acta matematika, published in Stockholm by Mittag – Leffler [1]. Later, it was included in his course of lectures, which were read at the Sorbonne and published in French [2].

Consider the problem of two species consuming the same food [2].

Suppose that with an amount of food sufficient to fully satisfy the species under consideration, there are constant positive growth coefficients $\varepsilon_1, \varepsilon_2$. In a real situation, when these species live in a limited area, food will decrease with increasing numbers N_1 and N_2 (which mean the number of species). This will lead to a decrease in the values of the growth coefficients. If the amount of food eaten per unit time is represented by the function $F(N_1, N_2)$, it turns into zero simultaneously with the sum $N_1 + N_2$ and monotonically approaches ∞ together with each of these variables, then it is natural to take the expressions as growth coefficients:

$$\varepsilon_1 - \gamma_1 F(N_1, N_2),\ \varepsilon_2 - \gamma_2 F(N_1, N_2), \tag{1}$$

where γ_1, γ_2 – positive constants corresponding to the food requirements of each of the two species.

From here we obtain a system of differential equations that describe the development of both species [2]:

$$\frac{dN_1}{dt} = \left[\varepsilon_1 - \gamma_1 F(N_1, N_2)\right] N_1, \tag{2}$$

$$\frac{dN_2}{dt} = \left[\varepsilon_2 - \gamma_2 F(N_1, N_2)\right] N_2. \tag{3}$$

Now the mathematical problem arises of studying the solutions N_1, N_2 of this system for initial values N_1^0, N_2^0, positive for $t = t_0$.

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It can be proved that for any finite time interval (t_0, T) there is a unique solution of two continuous functions, which are placed between two positive numbers, of which no longer depends on the end of the interval T (i.e. N_1, N_2 remain bounded).

Let us consider what happens with an unlimited increase in time. Rewriting (2) and (3) in the form:

$$\frac{d\log N_1}{dt} = \varepsilon_1 - \gamma_1 F(N_1, N_2), \tag{2-a}$$

$$\frac{d\log N_2}{dt} = \varepsilon_2 - \gamma_2 F(N_1, N_2), \tag{3-a}$$

we get:

$$\gamma_2 \frac{d\log N_1}{dt} - \gamma_1 \frac{d\log N_2}{dt} = \varepsilon_1\gamma_2 - \varepsilon_2\gamma_{1,} \tag{4}$$

and then:

$$\frac{N_1^{\nu_2}}{N_2^{\nu_1}} = \frac{\left(N_1^0\right)^{\nu_2}}{\left(N_2^0\right)^{\nu_1}} e^{(\varepsilon_1\gamma_2 - \varepsilon_2\gamma_1)(t-t_0)}. \tag{5}$$

We neglect the almost improbable case when:

$$\varepsilon_1\gamma_2 - \varepsilon_2\gamma_1 = 0, \tag{6}$$

and suppose (changing the types if necessary) that:

$$\varepsilon_1\gamma_2 - \varepsilon_2\gamma_1 > 0 \text{ або } \frac{\varepsilon_1}{\gamma_1} > \frac{\varepsilon_2}{\gamma_2}. \tag{7}$$

Then according to (5) we have:

$$\lim_{t\to 0} \frac{N_1^{\nu_2}}{N_2^{\nu_1}} = +\infty. \tag{8}$$

Since N_1 is bounded, N_2 tends to zero.

Thus, we can conclude *that the second species, in which ε/γ has a smaller value, will decrease and eventually disappear, while the first one continues to exist* [2].

Two species, one of which eats the other (predator and prey)

This problem has been solved by many researchers in the field of biology and medicine, in particular virology. Its partial solution is given in the book of A. Lotka [3], and a more general one in the lectures of V. Volterra [2]. Because of this, these equations are sometimes called the Lotka-Volterra equations [1].

If only one of them, namely the prey, were present in the environment where these species live, then it would have a certain growth coefficient ε_1, which we will assume to be constant and positive.

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The second species (predator), which feeds only (or mainly) on the prey, assuming that it exists in isolation, has a certain growth coefficient $-\varepsilon_2$, which we will assume to be constant and negative. When such two species exist in a limited environment, the first will develop the slower the more individuals of the second species exist, and the second – the faster the more numerous the first species is. The hypothesis, quite simple, is that the growth rates are equal to, respectively:

$$\varepsilon_1 - \gamma_1 N_2 \quad \text{and} \quad -\varepsilon_2 + \gamma_2 N_1 \quad , \tag{9}$$

(γ_1, γ_2 are positive constants). This leads to a system of differential equations for describing the number of species [26]:

$$\begin{aligned} \frac{dN_1}{dt} &= \left(\varepsilon_1 - \gamma_1 N_2\right) N_1, \\ \frac{dN_2}{dt} &= -\left(\varepsilon_2 - \gamma_2 N_1\right) N_2, \end{aligned} \qquad \left(\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2 > 0\right). \tag{10}$$

We will arrive at the same result with a less crude study of the interaction of species, reasoning as follows [2].

Let us consider the more general case of two species, which, existing separately, have growth coefficients λ_1, λ_2, the signs of which are not specified. In the case of coexistence of these species, we will assume that the meetings of individuals of different species (the number of these meetings per unit of time is equal to $\alpha N_1 N_2$, where $\alpha = const$) somehow affect the number of species. Algebraically, this effect is expressed by the increments β_1 and β_2 of the number of individuals corresponding to *n* meetings (*n* is a fixed, sufficiently large number). We will assume that these increments occur immediately, without delay. Then, over time *dt*, the species increase by:

$$dN_1 = \lambda_1 N_1 dt + \alpha N_1 N_2 \frac{\beta_1}{n} dt,$$

$$dN_2 = \lambda_2 N_2 dt + \alpha N_1 N_2 \frac{\beta_2}{n} dt.$$

Thus, we obtain a system of differential equations [2]:

$$\begin{aligned} \frac{dN_1}{dt} &= N_1 \left(\lambda_1 + \mu_1 N_2\right), \\ \frac{dN_2}{dt} &= N_2 \left(\lambda_2 + \mu_2 N_1\right), \end{aligned} \tag{11}$$

where:

$$\mu_1 = \alpha \frac{\beta_1}{n} \qquad \mu_2 = \alpha \frac{\beta_2}{n}. \tag{12}$$

Since encounters are beneficial for predators and detrimental for prey, in the case we will consider first,

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$$\lambda_1 > 0,\ \lambda_2 < 0,\quad \mu_1 \langle 0,\ \mu_2 \rangle 0. \tag{13}$$

Therefore, equations (11) takes the form (10) [2].
From equations (11) we obtain for any case (assuming $N_1 > 0, N_2 > 0$):

$$\mu_2 \frac{dN_1}{dt} - \mu_1 \frac{dN_2}{dt} = \mu_2\lambda_1 N_1 - \mu_1\lambda_2 N_2, \tag{14}$$

$$\lambda_2 \frac{\frac{dN_1}{dt}}{N_1} - \lambda_1 \frac{\frac{dN_2}{dt}}{N_2} = \mu_2\lambda_1 N_1 - \mu_1\lambda_2 N_2, \tag{15}$$

where:

$$\mu_2 \frac{dN_1}{dt} + \lambda_2 \frac{\frac{dN_1}{dt}}{N_1} - \mu_1 \frac{dN_2}{dt} - \lambda_1 \frac{\frac{dN_2}{dt}}{N_2} = 0. \tag{16}$$

Integrating, we obtain [2]:

$$\mu_2 N_1 + \lambda_2 \log N_1 - \left(\mu_1 N_2 + \lambda_1 \log N_2\right) = const, \tag{17}$$

Or

$$N_1^{\lambda_2} e^{\mu_2 N_1} = C N_2^{\lambda_1} e^{\mu_1 N_2}. \tag{18}$$

Let us construct the curve (18) in the plane (N_1, N_2).
Let us return to the case of predator and prey considered above, when

$$\lambda_1 = \varepsilon_1 > 0, \qquad \lambda_2 = -\varepsilon_2 < 0,\ \mu_1 = -\gamma_1 \langle 0, \quad \mu_2 = \gamma_2 \rangle 0. \tag{19}$$

To construct this curve, we will draw auxiliary curves

$$(L_1) \quad Y = N_1^{-\varepsilon_2} e^{\gamma_2 N_1}, \tag{20}$$

$$(L_2) \quad C = N_2^{\varepsilon_1} e^{\gamma_1 N_2}, \tag{21}$$

and draw the desired curve based on the relation

$$Y = CX. \tag{22}$$

On two perpendicular lines, we mark the axes Ox, ON_1 and iOy, ON_2 (Fig. 1), in the second and fourth quadrants we draw auxiliary curves L_1 and L_2.

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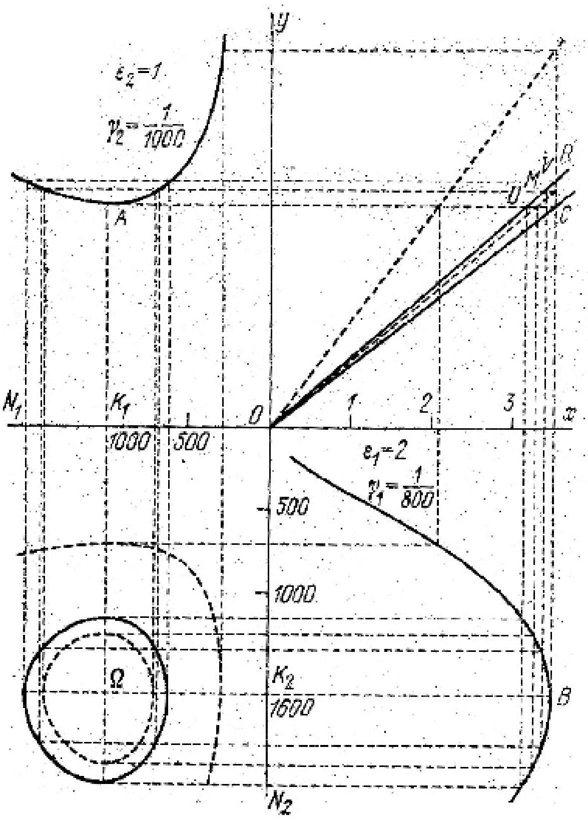

Fig. 1. Phase portraits of the predator-prey problem (Volterra, 1931)

In practice, this curve is a phase portrait of the "predator-prey" problem, on the basis of which we can consider and analyze various scenarios of this problem [7].

Adiabatic elimination of variables method

If in the system of equations

$$\dot{\vec{x}} = f\left(\vec{x},\vec{u},\vec{\xi},t\right), \tag{23}$$

the right-hand side does not depend explicitly on time, i.e. $\dot{\vec{x}} = = f\left(\vec{x},\vec{u},\vec{\xi}\right)$, then such a system of equations is called *autonomous* or *self-organized* (*self-controlled*) (Trokhimchuck, 2020). Here $\vec{x}$ is a deterministic phase vector, $\vec{u}$ is a vector of control parameters, $\vec{\xi}$ is a vector of stochastic parameters.

The most developed is the system of two nonlinear equations of the first order (on the plane) (Trokhimchuck (2020). It can always be represented as a Hamiltonian system and the entire arsenal of mathematical methods developed in this area can be applied.

Let us write the system of two equations in the following form:

$$\dot{x}_1 = f_1\left(x_1, x_2\right), \quad \dot{x}_2 = f_2\left(x_1, x_2\right). \tag{24}$$

System (24) will be non-autonomous if $\frac{\partial f_n}{\partial x}0$.

Let us consider the case when the variables x_1 and x_2 describe two different subsystems, slow and fast in time variation. For this case, equation (24) takes the form

$$\dot{x}_1 = k_{11} + F_1(x_1, x_2), \qquad \dot{x}_2 = -\gamma x_2 + F_2(x_1, x_2), \tag{25}$$

where $\gamma > 0,\ \gamma \gg |k_{11}|$. The evolution of a fast subsystem begins with a transient process of duration $\tau_2 \sim \gamma^{-1} \ll \tau_1 \sim |k_{11}^{-1}|$. Further, the evolution of the system is described by equations (8.2), in which we can put $\dot{x}_2 = 0$:

$$\dot{x}_1 = k_{11} + F_1(x_1, x_2), \qquad 0 = -\gamma x_2 + F_2(x_1, x_2). \tag{25-a}$$

From the second equation we find $x_2 = \varphi(x_1)$. Substitution x_2 into the first equation allows us to significantly simplify the problem:

$$\dot{x}_1 = k_{11} + F_1(x_1, \varphi(x_1)). \tag{26}$$

This method, based on the selection of characteristic time scales, is called *adiabatic elimination of the variable* x_2. This approach was first proposed by H. Haken [5]. It can be concluded that the behavior of the system is determined by the evolution of the slow subsystem. The slow subsystem controls the fast one. That is why the variable x_1 is called *the order parameter*.

In multidimensional systems, a small number of slow variables can be distinguished, to which all the others are adjusted. Moreover, in many cases, it is possible to obtain solutions of the form $x_n(t) = F(t, \psi(\zeta_n)),\ \ \zeta_n = {}^{n}\!/_{at},\ \ n \in (1, s)$. Such solutions are called self-similar, or self-similar. The evolution of the system is characterized by “forgetting” the initial conditions and the formation of structures determined by functions $\psi(\zeta_n)$. Simple structures are combined into various types of complex structures, to which the eigenvectors of a nonlinear system of equations can be compared. Such solutions cannot exist in the vicinity of the equilibrium state, since the dissipative process associated with the dissipation of energy destroys any order. New coherent structures arise in states far from equilibrium in open systems and are stabilized as a result of energy exchange with the environment. Thus, nonequilibrium can be a source of order, or self-organization. I. Prigozhin called such *order a dissipative structure* [6]. The phenomena of self-organization are inherent in hydrodynamics, chemistry, biology, astrophysics, ecology, economics, sociology. H. Haken proposed to call this part of the theory of control *synergetics* (literally – *the theory of joint action*) [5].

Since the systems of Volterra equations are autonomous equations, it is advisable to use the Haken procedure for them. Therefore, this method was tested for the system of three Volterra equations to describe the processes of Relaxed Optics with different adiabatic elimination procedures (single and double) [8–10].

Diffusion instability

Let us move on to a more complex example – a two-component system of the form:

$$\frac{\partial u_1}{\partial t} = f_1(u_1, u_2) + D_1 \frac{\partial^2 u_1}{\partial x^2}, \qquad \frac{\partial u_2}{\partial t} = f_2(u_1, u_2) + D_2 \frac{\partial^2 u_2}{\partial x^2}. \tag{27}$$

Where D_1 and D_2 are the diffusion coefficients. When $D_1 = D_2 = 0$, the system (27) reduces to a system (24) [1]. Therefore, system (27) may be represented as Volterra system with diffusion expansion too. Adding spatial derivatives, i.e. “turning on” the diffusion coupling of point elements, can radically change the properties of the system. Let us begin traditionally with finding and investigating the stability of the homogeneity of stationary states. According to (27), these states are determined from the system of equations:

$$f_1(u_1,u_2)=0, \quad f_2(u_1,u_2)=0. \tag{28}$$

Let $u_1=u_1^{(0)}$ and $u_2=u_2^{(0)}$ are some solutions of the system (28). To investigate the stability of these solutions, we write:

$$u_1(x,t)=u_1^{(0)}+w_1(x,t), \quad u_2(x,t)=u_2^{(0)}+w_2(x,t). \tag{29}$$

Substituting (29) into (27) and linearizing the resulting equations by small additions w_1 and w_2, we obtain a system of two linear partial differential equations with constant coefficients:

$$\begin{aligned} \frac{\partial w_1}{\partial t}&=f_{11}w_1+f_{12}w_2+D_1\frac{\partial^2 w_1}{\partial x^2}, \\ \frac{\partial w_2}{\partial t}&=f_{21}w_1+f_{22}w_2+D_2\frac{\partial^2 w_2}{\partial x^2}, \end{aligned} \tag{30}$$

where the derivatives $f_{ik}=\partial f_i / \partial u_k$ are calculated at $u=u^{(0)}$ the solution of system (30) is sought in the standard form:

$$w_k(x,t)=\bar{w}_k exp(pt+iqx), \; k=1,2. \tag{31}$$

Substituting (31) into (30) gives the dispersion equation:

$$p^2-\left[\bar{f}_{11}(q)+\bar{f}_{22}(q)\right]p+\bar{f}_{11}(q)\cdot\bar{f}_{22}(q)-\bar{f}_{12}(q)\cdot\bar{f}_{21}(q)=0, \tag{32}$$

where $\bar{f}_{11}(q)=f_{11}-q^2D_1$, $\bar{f}_{22}(q)=f_{22}-q^2D_2$. We emphasize that at $q=0$ the dispersion equation (32) coincides with the dispersion equation for the corresponding point subsystem. In other words, the values of the functions $p(q)$, determined by equation (32), at $q=0$ coincide with the Lyapunov exponents of the linearized point system [1].

Analysis of the dispersion equation (32) shows that the solutions of equation (27) $u_1=u_1^{(0)}$ and $u_2=u_2^{(0)}$, which are stable in the absence of transport processes, i.e., at $D_1=D_2=0$, may lose stability when diffusion is "turned on" [1]. From the point of view of the dispersion equation, this means that, although at $q = 0$ both of its roots are negative ($p_1(0) < 0$ and $p_2(0) < 0$), there is an interval of values of q in which at least one of the roots has a positive real part. The conditions for the existence of such an interval follow from equation (32). They are written in the form [1]:

$$\begin{cases} 1)\, f_{11}\cdot f_{22}-f_{12}\cdot f_{21}<0, \\ 2)\, f_{11}\cdot D_2-f_{22}\cdot D_1>0, \\ 3)\,(f_{11}\cdot D_2+f_{22}\cdot D_1)^2>4D_1\cdot D_2\,(f_{11}\cdot f_{22}-f_{12}\cdot f_{21}), \\ 4)\, f_{11}+f_{22}<0. \end{cases} \tag{33}$$

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From the above it follows that the occurrence of instability is due to the presence of transport processes. Therefore, such instability is called diffusion instability. Its important feature is the requirement of the difference of diffusion coefficients:

$$D_1 \neq D_2, \tag{34}$$

which follows from the comparison of the second and fourth inequalities in (33). Fig. 2 shows the dispersion curves $p_{1,2}(q)$ for the cases of absence and presence of diffusion instability.

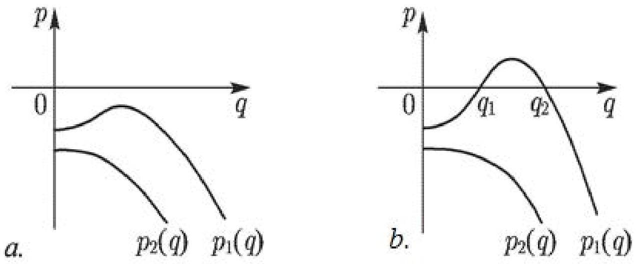

Fig. 2. Spectral functions *p* (*q*), given by the dispersion equation (14.71), for the stable (a) and unstable (b) homogeneous state (Trokhimchuck, 2020)

Conclusions

1. The systems of Volterra and Lotka-Volterra kinetic equations and their extensions are analyzed.
2. The problem of two species eating the same food is investigated. The conditions for the survival of one of the species are formulated.
3. The “predator-prey” problem is discussed. It is shown that for its analysis it is worth using the method of phase diagrams.
4. The problem of adiabatic elimination of variables for systems of autonomous differential equations is analyzed. It is shown that this procedure is also successfully used for systems of Volterra kinetic equations.
5. The problem of diffusion instability for systems of autonomous equations with the addition of diffusion terms is investigated. In general, in a particular case, this can be considered as a diffusion extension of the Volterra equations.

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